VOL5 NO. 1 March 2016 - ISSN 2233 – 1859

UOIuBIH
ORSinBIH
Operations Research Society in Bosnia and Herzegovina

Southeast Europe Journal of Soft Computing

Available online: http://scjournal.ius.edu.ba

IUS Soft Computing Research Group

Volatility Estimation Through Historical Prices of Indexes

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Article Info

Article history:
Article received on February 2016
Received in revised form on March 2016

Keywords:
Historical volatility, ARCH, GARCH, EWMA

Abstract

The volatility is the topic that has been researched in last few decades in various directions. One group of these methods includes the characteristics accumulated through the historical movements of the price of a specific instrument. Other than past data, most of them include other factors such as the stochastic part. In this paper revised are three methods of EWMA, ARCH, and GARCH.

1. INTRODUCTION

The main measure of an investment risk, whether of a single financial instrument (stock, bond, forward…) or a portfolio, would be the measure of its volatility. While considering the simplicity of calculating an implied volatility based on a given market price of an instrument, it is the volatility of the approaching period through which the instrument is to be held that actually matters.

In this paper, tested will be three different methods of using only the historical data to estimate the current volatility. Also, to make the comparison more conclusive, the calculation is being done using three sets of data, as to ensure the parallel result in each of them. Expected would be less extreme values, as all of the three sets of historical values of different Stock Indexes. Data used in this paper include two years of historical daily of the S&P100, S&P500, and DOW JONES INDUSTRIAL Indexes. So, volatility is being estimated for every day, and the daily estimation is being compared to the realized.

2. ESTIMATING VOLATILITY

It is also needed to mention that the formulation of ARCH as well as of GARCH used in this paper excludes the stochastic part. ARCH(m) would rather be of a form $\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i (\sigma_{n-i} z_t)$, $z_t \sim NID(0,1)$, where z_t is a random number under normal distribution with mean zero and standard deviation one (also known as white noise). The following structure, as used further in this paper, of both ARCH and GARCH was taken from the Hull(2009).

$$u_i = \ln \frac{S_i}{S_{i-1}}$$

The next step is to calculate the mean value of that continuous interest rate based on preceding m days by

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-1}.$$

The mean variance for a single day, calculated through those m values of the continuous interest rates of the preceding m number of days would be:

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2 \qquad 2.1$$

On the other hand, for its practicality, some considerably negligible changes are being made in the 2.1 formula. First of them is that instead of using the exponential formula for the continuous rate of interest, the u_i is to be calculated as a single day percentage change of the price

$$u_i = \frac{S_i - S_{i-1}}{S_{i-1}} \qquad 2.2$$

The mean value of interest rate $\bar{u}$ is considered to be of a negligible value and thus is being replaced by zero, having taken into consideration the same likelihood of positive and negative trends in price changes. And finally, the m-1 is being substituted by m. While making minimal if any change in the calculated values, these changes make it possible to reasonably simplify the formula for calculating the variance to:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2 \qquad 2.3$$

2.1 WEIGHTING SCHEMES

As seen in the equation 2.3 each value of u_i inclusively ranging from u_{n-m}, u_{n-m+1} onwards until u_{n-1} is being given the same weight of $\frac{1}{m}$. On the other hand, as to evaluate the current value, it would be more suitable to give higher weight to the more recent values. As to make that change in the above formula, the $\frac{1}{m}$ the part would be replaced by α_i which would be the weight assigned to the day $'n - i'$. That would provide us with a new formula:

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \qquad 2.4$$

The value of α_i is to be decreasing with the increase in i, that way the more recent the value is the more impact it is being assigned. At the same time the sum of all the α_i values are to add up to one.

$$\sum_{i=1}^{m} \alpha_i = 1$$

The 2.4 equation could be further expanded by giving some weight to a certain value of variance historical average, based on a history of that instrument. Taking V_L to be that long-term historical return variance; γ is the weight that is to be assigned to V_L. Now that the weight is to be distributed between γ and α_i, it provides:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \qquad 2.5$$

Where $\gamma + \sum_{i=1}^{m} \alpha_i = 1$

2.2 AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTICITY

In the mentioned formulas, the variance estimation is being divided between the historical long-run average variance and the weighted average of m preceding observations. Replacing the γV_L part of the formula with $\omega = \gamma V_L$ brings out the formula of autoregressive conditional heteroskedasticity (ARCH) originally suggested by Engle (1982):

$$\sigma_n^2 = \omega + \sum_{i=1}^{m} \alpha_i u_{n-1}^2 \qquad 2.6$$

Satisfaction with the outcomes achieved through ARCH has been reported by among others Diebold and Nerlove (1986), Lastrapes (1989). They have confirmed it suitable in different financial time-series.

In practice, in this paper the ARCH method was applied for S&P100 European Style Index also known as XEO, using two years of the daily closing price of the index. The daily percentage change was calculated using $u_i = \ln\frac{S_i}{S_{i-1}}$, and the formula 2.6 was used to, through optimization, minimize the squared difference between ARCH volatility estimation and the CBOE S&P 100 Volatility Index (VXO) by changing the values of w and α_i.

Among the considerations was how many periods ‘ p’ ARCH(p) is to be considered. While solving with small number of α_i's the optimization results were not stable, rather through running the same process again and again it would give a number of different ‘optimization’ results. On the other hand, having i>10 did show stability in the answers as did that same increase improve the optimization (minimization of the difference between ARCH(p) and VXO values). The following were the optimization results for w and α_i for i=15

S&P100 European Style XEO	S&P500 European Style SPX	Dow Jones Industrial Average DJI
w = 5.67E-05	w = 5.51E-05	w = 5.13E-05
α1 = 0.129687	α1 = 0.119323	α1 = 0.135967
α2 = 0.086247	α2 = 0.083616	α2 = 0.092107
α3 = 0.057773	α3 = 0.061001	α3 = 0.061356
α4 = 0.029855	α4 = 0.02928	α4 = 0.033398
α5 = 0.003982	α5 = 0.001661	α5 = 0.007507
α6 = 0.036741	α6 = 0.036785	α6 = 0.041574
α7 = 0.032502	α7 = 0.036212	α7 = 0.035127
α8 = 0.020817	α8 = 0.029537	α8 = 0.023854
α9 = 0.008472	α9 = 0.020668	α9 = 0.010616
α10 = 0.003079	α10 = 0.007451	α10 = 0.010671
α11 = 0.014713	α11 = 0.022038	α11 = 0.020176
α12 = 0	α12 = 0	α12 = 0
α13 = 0	α13 = 0	α13 = 0
α14 = 0	α14 = 0	α14 = 0
α15 = 0	α15 = 0	α15 = 0

Then, upon testing the forecast based on the above formula and the data, the mean value of the Squared Differences was 0.000505. The mean value of the square root of the Squared Differences was 0.0169 while its median was 0.0139. Maximum difference was 0.152 while the minimum difference was 0.00005. The Standard Deviation of the difference between volatility forecast through ARCH(15) and the VXO was 0.01848 or 1.85%. The maximum difference of 0.152 is 9.14 standard deviations above the mean.

One of the characteristics of the outcomes the ARCH gave, as opposed to initial expectation, is that the weight was not decreasing orderly through increase in i of corresponding α_i. Instead, it was decreasing from i=1 till i=5, and then after noticeable increase for i=6 decreasing until i=10 (inclusively), and then from i=10, it would start increasing but starting from i=12 on $\alpha_i = 0$. Parallel is the movement, increase-decrease, for DJI, as well as for SPX ARCH(15), where in addition to α_i for i=5 and i=10 having values of local minimum, and starting from i=12 and on all α_i have the value of zero.

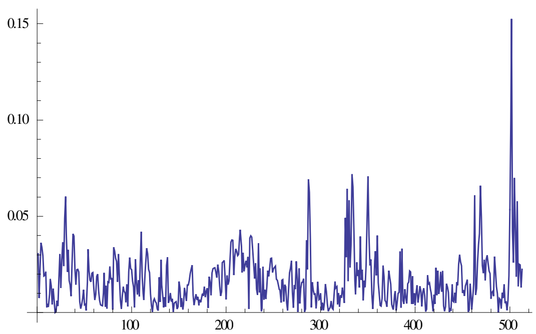

Figure 2.1 – Squared difference between XEO ARCH(15) outcomes and VXO

As outcome in Figure 2.1 shows that good majority of XEO volatility estimations have the squared difference below 0.002. Actually, 79.72% of ARCH(15) estimations had error below 0.025 or 2.5% (corresponding to square error of 0.000625). On the other hand, 99.6% of ARCH(15) had error below 0.1 or 10% (or squared error of 0.01).

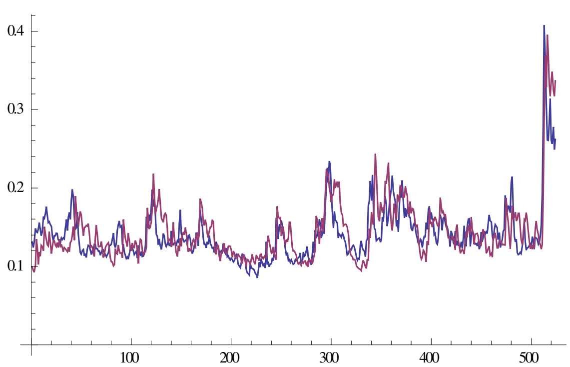

Figure 2.2 – The XEO ARCH(15) outcomes and VXO

Figure 2.2 shows the VXO (CBOE S&P 100 Volatility Index), and the ARCH(15) volatility estimation for S&P100 European Style Index. It can be seen that at extreme changes of the market volatility, ARCH(15) needs a few trading periods to catch that change, which could be considered as a weakness point of taking the past fifteen periods of price change.

Doing the ARCH(4) for the same data of S&P100 European Style Index (XEO), there was not a big underperformance considering much less of historical data used in each calculation. The same conclusion can more assertively be concluded through the mean value of the Squared Differences 0.000545 as opposed to 0.000505 that occurred in ARCH (15).

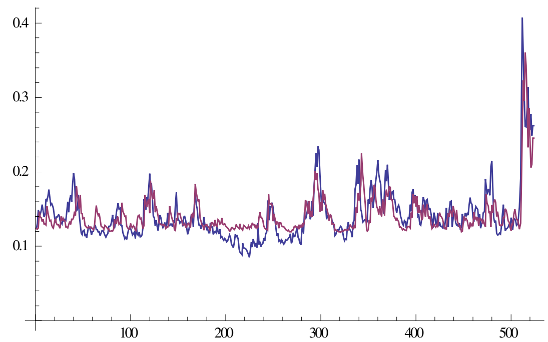

Figure 2.2 – The XEO ARCH(4) outcomes and VXO

The ARCH was further developed by Engle, Granger and Kraft (1984), as they were the first to publish an extension of the ARCH model to be multivariate. Besides that, Bera and Higgins (1993) state a few advantages of the ARCH as the main grounds for its success. According to them, while managing the clustered errors as much as it does nonlinearities, ARCH models are simple and easy to handle.

2.3 EXPONENTIALLY WEIGHTED MOVING AVERAGE MODEL

When applying the EWMA there is no chance that an older historical value gets to be given higher weight, as was the case in the outcomes of previous three examples in ARCH. In the Exponentially Weighted Moving Average (EWMA) Model, the formula is being developed based on the equation 2.5, where the historical long-run average variance V_L is being replaced by a single period volatility. At the same time, while looking back for the data of m periods, it includes an exponential decline of the weight given to previous returns u_{n-i} respectively with an increase in i.

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

The difference is that the exponential change of weight is based on one single value λ which when compared to ARCH replaces the two variables of γ & α, while its possible value ranges between 0 and 1. So, the variance σ_n^2 of the day 'n' can be estimated based on the variance of the preceding day σ_{n-1}^2. Similarly, taking an additional step back for one period, the σ_{n-1}^2 would be calculated based on σ_{n-2}^2 as:

$$\sigma_{n-1}^2 = \lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2$$

By putting the same into the σ_n^2 equation, the extended formula becomes:

$$\sigma_n^2 = \lambda(\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2) + (1-\lambda)u_{n-1}^2$$

With few more algebraic steps it becomes:

$$\sigma_n^2 = (1-\lambda)(u_{n-1}^2 + \lambda u_{n-2}^2) + \lambda^2\sigma_{n-2}^2$$

Applying the previous steps by substituting for σ_{n-2}^2 makes it:

$$\sigma_n^2 = (1-\lambda)(u_{n-1}^2 + \lambda u_{n-2}^2 + \lambda^2 u_{n-3}^2) + \lambda^3\sigma_{n-3}^2$$

Going on in the same direction provides

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-1}^2 + \lambda^m\sigma_{n-m}^2 \qquad 2.7$$

Where, knowing that λ ranges between 0 and 1, with higher values of m, the value of $\lambda^m\sigma_{n-m}^2$ becomes reasonably small, mostly due to λ^m, so it can be overlooked. So the remaining

$$\sigma_n^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-1}^2$$

Becomes just the same as the equation 2.4, where the value of each α_i would be substituted by $(1-\lambda)\lambda^{i-1}$.

In practice EWMA is being used by the JPMorgan's RiskMatrics database, which was providing the daily volatility estimates, having determined the best value of λ to be 0.94. Having it applied with the data used previously for ARCH method, slightly different results were attained.

The question is how to determine the best value of the λ, as well as the number of periods to go back in the calculation process "m".

2.4 THE GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSKEDASTICITY

The GARCH(p, q) model was originally introduced by Bollerslev(1986), as a further development of the Engle's ARCH. In GARCH(p, q), in addition to the long-run average variance rate, V_L, and the past periodical percentage change of the price, u_i, the past periodical volatility value is also taken into consideration. That periodic volatility would also be combined with a stochastic variable, but as stated before, the stochastic part is not included in this paper.

As GARCH(p,q) would look back for the price percentage change for p periods, it would look into q past volatility values. So looking only at the single period percentage change, as well as a single period volatility, achieved would be the GARCH(1,1), the model suggested by Taylor (1986), where

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 \qquad 2.8$$

Where γ, α, and β are the weights assigned to V_L, u_{n-1}^2 and σ_{n-1}^2, respectively. Those weights must add up to a total of 1:

$$\gamma + \alpha + \beta = 1$$

In this case, the previously covered model of Exponentially Weighted Moving Average (EWMA) would be a specific case of GARCH(1,1) where $\gamma = 0, \alpha = 1 - \lambda$, and $\beta = \lambda$.

As done in the ARCH model, the γV_L part of the formula is being replaced with ω:

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2 \qquad 2.9$$

The above formula is being used as to evaluate the best distribution of the parameters w, α, and β.The value of γ can thereby be calculated as $\gamma = 1 - \alpha - \beta$, and thus follows the calculation of $V_L = \frac{\omega}{\gamma}$. Reasonably, the sum of α and βmust be less than 1, as otherwise γ would be negative.

As for the GARCH(p, q), equation is:

$$\sigma_n^2 = \omega + \sum_{i=1}^{q}\alpha_i u_{n-i}^2 + \sum_{j=1}^{p}\beta_j\sigma_{n-j}^2 \qquad 2.10$$

Taking the formula of GARCH(1,1), and replacing its σ_{n-1}^2 part of $\beta\sigma_{n-1}^2$ with its GARCH(1,1) gives

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta(\omega + \alpha u_{n-2}^2 + \beta\sigma_{n-2}^2)$$

$$\sigma_n^2 = \omega + \beta\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \beta^2\sigma_{n-2}^2$$

Repeating the same procedure, this time for σ_{n-2}^2

$$\sigma_n^2 = \omega + \beta\omega + \beta^2\omega + \alpha u_{n-1}^2 + \alpha\beta u_{n-2}^2 + \alpha\beta^2 u_{n-3}^2 + \beta^3\sigma_{n-2}^2$$

Proceeding the same way brings the final form of

$$\sigma_n^2 = \omega + \sum_{i=1}^{k}\beta^i\omega + \sum_{i=0}^{k}\alpha\beta^i u_{n-i-1}^2 + \beta^{k+1}\sigma_{n-k}^2$$

Taking a reasonably high value of k would make the value of β^{k+1} as low as to eliminate the significance of past value of volatility σ_{n-k}^2, just as is the case in EWMA. The significant difference remaining between the two is the weight GARCH(1,1) gives to the historical volatility of V_L $(\omega = \gamma V_L)$.

2.5 THE MAXIMUM LIKELIHOOD METHOD

The question remaining is how to determine the values of α, β, and γ $(w = \gamma V_L)$ for the GARCH(p,q), as well as the value of λ for the EWMA model. The first assumption is that the price change between two periods is normally distributed with the zero mean and the variance is to be calculated. Assume having m observations with each having a single period percentage change of $u_1, u_2, \dots, u_m$. The probability density function formula gives the probability of occurrence of u_i

$$\frac{1}{\sqrt{2\pi\upsilon}}\exp\left(\frac{-u_i^2}{2\upsilon}\right)$$

The variable υ is the variance value that is to be estimated. Probability of occurrence of these m observations, in the observed order, would be the product of the probability of each

$$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi\upsilon}}\exp\left(\frac{-u_i^2}{2\upsilon}\right)\right] \quad 2.11$$

The estimation of the variance value υ is to be done as to maximize the value of this expression. Maximum of an expression is the same as maximization of its Natural Logarithm. Moreover, constant multiplicative factors can be ignored, as they do not influence the maximization process. So taking the natural log of the above formula gives

$$\sum_{i=1}^{m}\left[\ln\left(\frac{1}{\upsilon}\right)-\frac{u_i^2}{\upsilon}\right] \quad 2.12$$

This can be further rewritten as

$$-m\ln(\upsilon)-\sum_{i=1}^{m}\frac{u_i^2}{\upsilon}$$

Taking its first derivative with respect to υ, and assigning it to zero value, in order to reach the maximum likelihood estimation of

$$\frac{1}{m}\sum_{i=1}^{m}u_i^2$$

Where in both cases of EWMA or GARCH(1,1), the single value of υ, is to be replaced by the periodical value, which would be the estimated variance by the method. Maximization is to be done to the

$$\sum_{i=1}^{m}\left[\ln\left(\frac{1}{\upsilon_i}\right)-\frac{u_i^2}{\upsilon}\right]$$

Or rewrite it as

$$\sum_{i=1}^{m}\left[-\ln\upsilon_i-\frac{u_i^2}{\upsilon}\right] \quad 2.13$$

3. APPLICATION AND COMPARISON

Let EWMA(m) be the EWMA calculation based on m most recent observations. The calculation is done by assigning the above objective function (2.13) to be maximized, where $u_i=\ln\frac{S_i}{S_{i-1}}$, and the calculated variance estimation for each period of

$$\upsilon_i=(1-\lambda)\sum_{i=1}^{m}\lambda^{i-1}u_{n-1}^2+\lambda^m\sigma_{n-m}^2.$$

In the following examples, the same XEO data of 525 trading days were used. Calculating EWMA(25) gave the weight value of λ =0.8797. On the same data, EWMA(10) gave the weight value of 0.8982. The two also gave a very similar Total Sum of the Maximization Formula of 4598.18 and 4595.62 respectively, which shows that the additional 15 periods used in EWMA did not make a significant difference in the outcome.

Similar optimization was done for GARCH(1,1), where $u_i=\ln\frac{S_i}{S_{i-1}}$,and $\upsilon_n=\omega+\alpha u_{n-1}^2+\beta\sigma_{n-1}^2$, where all three variables of $\omega,\alpha,$ and β are to be determined through optimization. The expression $(-\ln\upsilon_i-\frac{u_i^2}{\upsilon})$ was calculated for every trading day, and added up for the 525 trading days of the sample, where the objective is to maximize the sum. The values assigned to the variables were ω =0.0000791, α =0.193225, and $\beta=$ 0.67974 while the total sum that was maximized reached the value of 4618.45.

Interestingly, GARCH(1,1) outperforms EWMA(25), even though the difference is not significant. An interesting conclusion is that a single period GARCH outperforms 25 or 10 periods EWMA. It might still be questionable whether it always outperforms it, given the difference being “small”.

Similarly, the calculation was done for S&P500 Index (SPX) where EWMA(25) gave the maximized sum of 4591.88, whereas its GARCH(1,1) returned the sum of 4618.42. The same was done for the index of Dow Jones Industrial (DJI), and the maximized sum by EWMA(25) of 4611.66 was clearly even closer to that in GARCH(1,1) of 4626.52.

3.1 Comparing the outcomes of GARCH and ARCH

When comparing the outcomes of the same GARCH(1,1) with the ARCH(15), the mean square difference previously calculated for ARCH(15) is 0.000762, whereas for the GARCH(1,1) the same difference is 0.00109. This outcome suggests ARCH(15) surpasses GARCH(1,1).

On the other hand, knowing that GARCH(1,1) was optimized without the use of VIX data, as opposed to ARCH(15) while the Error is calculated based on that same VIX data. The similar negative difference for GARCH(1,1) was reached with SPX and DJI, giving MSE of 0.001092 and 0.001093 respectively.

On the other hand, when repeating the optimization for GARCH(1,1), having replaced the objective function (2.13), used in the previous optimization, with the square difference between the GARCH(1,1) and the CBOE S&P 100 Volatility Index (VXO). In this case, GARCH (1,1) outcomes change to ω =0.0000124, α =0.1158 and β =0.7602 slightly outperforming ARCH(15), with the average mean squared difference being 0.000499.

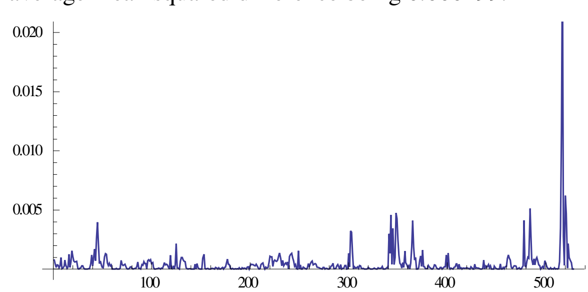

Figure 3.1 – Squared difference between XEO GARCH(1,1) outcomes and VXO

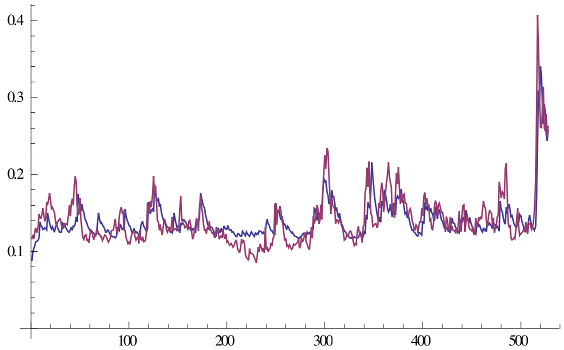

Figure 3.2 – Volatility XEO GARCH(1,1) outcomes and VXO

As can be seen in the two figures 3.1 and 3.2, the extreme cases of squared difference exceeding 0.01 occur very rarely, twice in the two years of the daily forecast. It is also the case that the two are two trading periods in a row (24th and 25th of August 2015), which would probably be days of some market.

When compared to the outcomes of its estimations, 79.7% of the GARCH forecasts were having an error of 2.5% or less, and at the same time, 99.6% of its estimations had an error exceed 10%. In this context, very comparable to ARCH(15).

Doing the calculation for DJI for the same period of 525 trading days the GARCH(1,1) returns mean square difference of 0.000340193, whereas the same data done by ARCH(15) had MSD of 0.00045. In parallel SPX data gave a mean square difference of 0.000501 under GARCH(1,1), while its ARCH(15) provided a somewhat better forecast with MSD of 0.0004993.

4. CONCLUSION

Observing ARCH outcomes for all three Indexes, it can be concluded that ARCH(11) would have been totally equivalent since the weight of zero was given to the last four periods. Another question is about i=5 and i=10, does it correspond to five working/trading days per week (with few exceptions of holidays)?

Revising the outcomes of all three EWMA, ARCH, and GARCH, while having in mind to using 25 past periods of data for EWMA and 15 past periods of data for ARCH on one hand, and one single period for GARCH, in the majority of cases the GARCH turned out to outperform both. The outcome was the same in each of S&P100 (XEO), and DOW JONES, with the small exception of S&P500 (SPX), where ARCH(15) did better than GARCH(1,1).

The ARCH model should be revisited, given that the weight was not orderly in its changes with time, but rather followed a specific sequence. In addition to that, the next step will be to apply some different optimization methods for ARCH(p) and GARCH(p, q) and accordingly compare their outcomes.

REFERENCES

Bera, A. K., & Higgins, M. L. (1993). ARCH models: properties, estimation and testing. Journal of economic surveys, 7(4), 305-366.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. Journal of econometrics, 31(3), 307-327.

Bollerslev, T., Engle, R. F., & Wooldridge, J. M. (1988). A capital asset pricing model with time-varying covariances. The Journal of Political Economy, 116-131.

Diebold, F. X., & Nerlove, M. (1989). The dynamics of exchange rate volatility: a multivariate latent factor ARCH model. Journal of Applied econometrics, 4(1), 1-21.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. Econometrica: Journal of the Econometric Society, 987-1007.

Engle, R. F., Granger, C. W., & Kraft, D. (1984). Combining competing forecasts of inflation using a bivariate ARCH model. Journal of economic dynamics and control, 8(2), 151-165.

French, K. R., & Roll, R. (1986). Stock return variances: The arrival of information and the reaction of traders. Journal of financial economics, 17(1), 5-26.

Hull, J. (2009). Options, Futures and Other Derivatives. Pearson Education.

Lastrapes, W. D. (1989). Exchange rate volatility and US monetary policy: an ARCH application. journal of Money, Credit and Banking, 21(1), 66-77.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. Journal of financial economics, 8(4), 323-361.

Pindyck, R. S. (1984). Risk, Inflation, and the Stock Market. American Economic Review, 74(3), 335-51.

Poterba, J. M., & Summers, L. H. (1986). The Persistence of Volatility and Stock Market Fluctuations. The American Economic Review, 76(5), 1142-1151.

Taylor, S. (2000). (1986), Modelling Financial Time Series.