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On the Upper Bound for Growth Rate of Hydromagnetic Swirling- flows

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ABSTRACT: We consider stability of inviscid, incompressible, hydromagnetic swirling flows. We obtained supremum for the growth rates. Growth rate has been illustrated with three standard examples. Growth rate depends up on vorticity function, velocity profile and wave number. Furthermore, we obtained upper and lower bound for neutral phase speed. Also, we derived an instability regions depending on Rayleigh-Synge-Michael discriminant, velocity profile and radii.

Keywords:

Swirling flow, Co-axial Cylinders, Incompressible, Inviscid.

1. INTRODUCTION

We consider hydromagnetic coaxial flows which is inviscid and incompressible. When magnetic parameter becomes zero, it leads to Circular Rayleigh problem. Batchelor & Gill (1962) [cf. 1] obtained a condition for instability. Batchelor & Gill (1962) [cf. 1] obtained a semi-circle instability region. Anil & Subbiah (2010) [cf. 6] obtained bounds for neutral phase speed. Anil & Subbiah (2010) [cf. 6] obtained sufficient condition for stability. Pavithra & Subbiah (2021) [cf. 7] derived parabolic instability regions under some condition. Chandrashekhar et al (2022) [cf. 3] obtained instability region that intersect with Batchelor & Gill semi-circle under certain aproximation. Chandrashekhar et al (2023) [cf. 4] obtained a condition for stability and obtained instability which intersect with Batchelor & Gill semicircle.
For hydromagnetic swirling flows Sasakura (1984) [cf. 10] obtained a condition for stability. Sasakura (1984) [cf. 10] derived a Semi-circle region which depends on Richardson number. Prakash & Subbiah (2021) [cf. 8] derived instability regions under conditions. Anil & Subbiah (2010) [cf. 6] obtained a semielliptical region under condition that minimum curvature should be positive. Prakash & Subbiah (2021) [cf. 8] obtained estimate for growth rate. Chandrashekhar et al (2023) [cf. 5] obtained a condition for wave number for stability.

In this paper, we derived supremum estimate for growth rates. We obtained upper and lower bound for neutral phase speed. Also, we derived instability regions which depends on basic velocity profile, minimum Richardson number, curvature and Wave number.

2. HETEROGENEOUS STABILITY EQUATION

The heterogeneous the differential equation is given by (see [10, 11])

$$D\left(\frac{\rho_0 D(ru)}{r}\right)-k^2\rho_0 u-\frac{rD\left(\frac{\rho_0 DW}{r}\right)u}{W-c}$$

$$+\frac{\psi u}{(W-c)^2}=0, \qquad (1)$$

with boundary conditions

$$u(R_1)=0=u(R_2). \qquad (2)$$

Where k is the wave number, c complex eigen value, R_1, R_2 are the radial positions of the cylindrical walls,

$\psi=\frac{\rho_0\, D\left(r^2\Omega_0\right)^2}{r^3}+\left(D\rho_0\right)r\Omega_0{}^2-r\mu D\left(\frac{H_0^2}{r^2}\right)$ is the Rayleigh-Synge-Michael discriminant, $\Omega_0=\frac{V}{r}$ angular velocity and μ magnetic permeability.

3. GROWTH RATE

3.1 Theorem *The upper bound for growth rate is*

$$\left(kc_i\right)^4 \le \left|\psi^2\right|_{max}-\left|\psi\right|_{\min}\left|r\,D\left(\frac{\rho_0\,DW}{r}\right)\right|_{\min}\left|\frac{W_{max}-W_{min}}{2}\right|$$
$$+R_2\left|D\left(\frac{\rho_0\,DW}{r}\right)\right|^2_{\max}\left|\frac{W_{max}-W_{min}}{2}\right|.$$

Proof: Multiplying (1) by $rD\left(\frac{\rho_0 D\left(ru^*\right)}{r}\right)$ integrating over $\left(R_1, R_2\right)$ and applying (2), we have

$$\int_{R_1}^{R_2}\left|D\left(\frac{\rho_0 D(ru)}{r}\right)\right|^2 r\,dr+k^2\int_{R_1}^{R_2}\rho_0\frac{\left|D(ru)\right|^2}{r}r\,dr$$
$$+\int_{R_1}^{R_2}\left[\frac{\psi}{(W-c)^2}-\frac{rD\left(\frac{\rho_0 DW}{r}\right)}{W-c}\right]urD\left(\frac{\rho_0 D\left(ru^*\right)}{r}\right)dr=0. \qquad (3)$$

From (1), considering complex conjugate, we have

$$D\left(\frac{\rho_0 D\left(ru^*\right)}{r}\right)=\left[k^2\rho_0+\frac{rD\left(\frac{\rho_0\,DW}{r}\right)}{W-c^*}-\frac{\psi}{\left(W-c^*\right)^2}\right]u^*$$
$$. \qquad (4)$$

Sub. (4) in (3), equating real parts, we get

$$\int_{R_1}^{R_2}\left|D\left(\frac{\rho_0 D(ru)}{r}\right)\right|^2 r\,dr+k^2\int_{R_1}^{R_2}\rho_0\frac{\left|D(ru)\right|^2}{r}r\,dr$$
$$+k^2\int_{R_1}^{R_2}\frac{\rho_0\psi\left[\left(W-c_r\right)^2-c_i^2\right]}{\left|W-c\right|^4}r\left|u\right|^2 dr$$
$$+2\int_{R_1}^{R_2}\frac{\psi rD\left(\frac{\rho_0 DW}{r}\right)\left(W-c_r\right)}{\left|W-c\right|^4}\left|u\right|^2 dr-\int_{R_1}^{R_2}\frac{\psi^2}{\left|W-c\right|^4}r\left|u\right|^2 dr$$
$$-k^2\int_{R_1}^{R_2}\frac{\rho_0 D\left(\frac{\rho_0 DW}{r}\right)\left(W-c_r\right)}{\left|W-c\right|^2}r\left|u\right|^2 dr-\int_{R_1}^{R_2}\frac{\left[rD\left(\frac{\rho_0 DW}{r}\right)\right]^2}{\left|W-c\right|^2}\left|u\right|^2 dr=0. \qquad (5)$$

Multiplying eq. (1) by $\left(ru^*\right)$, integrating, using (2) and equating real parts, we have

$$\int_{R_1}^{R_2}\rho_0\left[\frac{\left|D(ru)\right|^2}{r}\right]dr+k^2\int_{R_1}^{R_1}\rho_0 r\left|u\right|^2 dr+\int_{R_1}^{R_1}\frac{r\,D\left(\frac{\rho_0\,DW}{r}\right)\left(W-c_r\right)}{\left|W-c\right|^2}r\left|u\right|^2 dr$$
$$-\int_{R_1}^{R_1}\frac{\psi\left[\left(W-c_r\right)^2-c_i{}^2\right]}{\left|W-c\right|^4}r\left|u\right|^2 dr=0. \qquad (6)$$

Multiplying (6) by k^2 and adding (5), we have

$$\int_{R_1}^{R_2}\left|D\left(\frac{\rho_0 D(ru)}{r}\right)\right|^2 r\,dr+2k^2\int_{R_1}^{R_2}\rho_0\frac{\left|D(ru)\right|^2}{r}r\,dr$$
$$+k^4\int_{R_1}^{R_2}\left|u\right|^2 r\,dr+2\int_{R_1}^{R_2}\frac{\psi D\left(\frac{\rho_0 DW}{r}\right)\left(W-c_r\right)}{\left|W-c\right|^4}r\left|u\right|^2 dr$$
$$-\int_{R_1}^{R_2}\frac{\psi^2}{\left|W-c\right|^4}r\left|u\right|^2 dr-\int_{R_1}^{R_2}\frac{\left[rD\left(\frac{\rho_0 DW}{r}\right)\right]^2}{\left|W-c\right|^2}\left|u\right|^2 dr=0. \qquad (7)$$

In equation (7), the first two terms are non-negative, dropping them, we get

$$k^4\int_{R_1}^{R_2}|u|^2 r\,dr+2\int_{R_1}^{R_2}\frac{\psi D\left(\frac{\rho_0 DW}{r}\right)(W-c_r)}{|W-c|^4}\,r|u|^2\,dr$$

$$-\int_{R_1}^{R_2}\frac{\psi^2}{|W-c|^4}\,r|u|^2\,dr-\int_{R_1}^{R_2}\frac{\left[rD\left(\frac{\rho_0 DW}{r}\right)\right]^2}{|W-c|^2}\,|u|^2\,dr=0.$$

Using triangular inequalities

$$\frac{1}{|W-c|^2}\le\frac{1}{c_i^{\,2}},\ 2(W-c_r)c_i\le|W-c|^2$$

and $c_i\le\left[\frac{W_{\max}-W_{\min}}{2}\right]$, we have

$$(kc_i)^4\ \le\ \left|\psi^2\right|_{max}-|\psi|_{\min}\left|rD\left(\frac{\rho_0 DW}{r}\right)\right|_{\min}\left|\frac{W_{max}-W_{min}}{2}\right|$$

$$+R_2\left|D\left(\frac{\rho_0 DW}{r}\right)\right|^2_{\max}\left|\frac{W_{max}-W_{min}}{2}\right|. \qquad (7)$$

3.1 Example

Let $W=r,\ r\,\varepsilon\,[1,2]$ and $\psi(r)=r.$

After calculations we get $kc_i\le 2.08$ for $\rho_0=1.$

3.2 Example

Let $W=\sin r,\ r\,\varepsilon\,[1,2]$ and $\psi(r)=r.$

After calculations we get $kc_i\le 1.42$ for $\rho_0=1.$

3.3 Example

Let $W=4r(r-1),\ r\,\varepsilon\,[1,2]$ and $\psi(r)=r.$

After calculations we get $kc_i\le 3.34$ for $\rho_0=1.$

3.2 Theorem *The upper bound for growth rate is*

$$(kc_i)^4\ \le\ \frac{\left|\psi^2\right|_{max}-|\psi|_{min}\left|rD\left(\frac{\rho_0 DW}{r}\right)\right|_{min}\left|\frac{W_{max}-W_{min}}{2}\right|+R_2\left|D\left(\frac{\rho_0 DW}{r}\right)\right|^2_{max}\left|\frac{W_{max}-W_{min}}{2}\right|}{\left[\left((\rho_0)_{min}\frac{R_1\pi^2}{k^2R_2(R_2-R_1)^2}\right)\left(\frac{\pi^2}{k^2(R_2-R_1)^2}+2\right)+1\right]}.$$

Proof:

In equation (7), applying Rayleigh-Ritz inequality, we get

$$\left[(\rho_0)_{\min}\frac{R_1\pi^4}{R_2(R_2-R_1)^4}+(\rho_0)_{\min}\frac{R_1\pi^2(2k^2)}{R_2(R_2-R_1)^2}+k^4\right]$$

$$.\int_{R_1}^{R_2}|u|^2 r\,dr+2\int_{R_1}^{R_2}\frac{\psi D\left(\frac{\rho_0 DW}{r}\right)(W-c_r)}{|W-c|^4}\,r|u|^2\,dr$$

$$-\int_{R_1}^{R_2}\frac{\psi^2}{|W-c|^4}\,r|u|^2\,dr-\int_{R_1}^{R_2}\frac{\left[D\left(\frac{\rho_0 DW}{r}\right)\right]^2}{|W-c|^2}\,r^2|u|^2\,dr\le 0.$$

Using inequalities

$$\frac{1}{|W-c|^2}\le\frac{1}{c_i^{\,2}},\ 2(W-c_r)c_i\le|W-c|^2 \text{ and}$$

$c_i\le\left[\frac{W_{\max}-W_{\min}}{2}\right]$, we have

$$(kc_i)^4\ \le\ \frac{\left|\psi^2\right|_{max}-|\psi|_{min}\left|rD\left(\frac{\rho_0 DW}{r}\right)\right|_{min}\left|\frac{W_{max}-W_{min}}{2}\right|+R_2\left|D\left(\frac{\rho_0 DW}{r}\right)\right|^2_{max}\left|\frac{W_{max}-W_{min}}{2}\right|}{\left[\left((\rho_0)_{min}\frac{R_1\pi^2}{k^2R_2(R_2-R_1)^2}\right)\left(\frac{\pi^2}{k^2(R_2-R_1)^2}+2\right)+1\right]}.$$

3.4 Example Let $W=r,\ r\,\varepsilon\,[1,2]$ and $\psi(r)=r.$

After calculations we get

$$kc_i\le\frac{2.08}{\left[\frac{\pi^4}{4k^4}+\frac{\pi^2}{k^2}+1\right]^{\frac14}}\quad\text{for } \rho_0=1.$$

3.5 Example Let $W=\sin r,\ r\,\varepsilon\,[1,2]$ and $\psi(r)=r.$

After calculations we get

$$kc_i\le\frac{1.42}{\left[\frac{\pi^4}{4k^4}+\frac{\pi^2}{k^2}+1\right]^{\frac14}}\quad\text{for } \rho_0=1.$$

3.6 Example Let $W=4r(r-1),\ r\,\varepsilon\,[1,2]$

and $\psi(r)=r$.

After calculations we get

$$kc_i \le \frac{3.34}{\left[\frac{\pi^4}{4k^4}+\frac{\pi^2}{k^2}+1\right]^{\frac{1}{4}}} \quad \text{for } \rho_0=1.$$

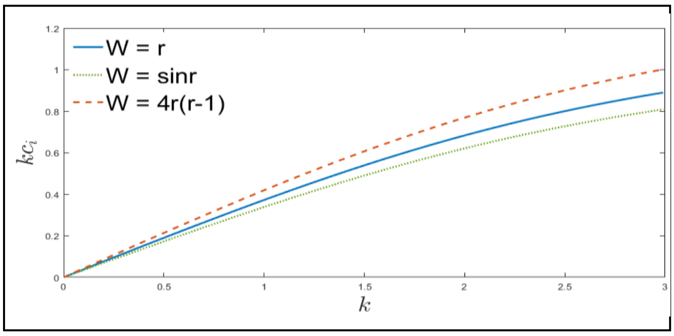

Figure 3.1: k vs kc_i (Growth rates for various velocity profiles)

4. BOUNDS FOR NEUTRAL PHASE SPEED

4.1 Theorem *The upper and lower bound of neutral phase speed is*

$$R_1 W_{min} - \frac{2|\psi|_{max}}{\left|r\,D\left(\frac{\rho_0 W}{r}\right)\right|_{\min}} \le c_r \le W_{max}.$$

Proof:

Multiplying (1) by (ru^*), integrating between R_1 and R_2, using (2) and taking imaginary parts, we have

$$\int_{R_1}^{R_1}\left[\frac{r\,D\left(\frac{\rho_0 DW}{r}\right)|W-c|^2-2\psi\left(W-c_r\right)}{|W-c|^4}\right] r|u|^2\,dr=0.$$

For real eigen values, $c=c_r$, we have

$$r\,D\left(\frac{\rho_0 DW}{r}\right)c_r^2-2\left(D\left(\frac{\rho_0 DW}{r}\right)rW-\psi\right)c_r+\left[rD\left(\frac{\rho_0 DW}{r}\right)W^2-2\psi W\right]=0.$$

Solving for c_r, we get

$$R_1 W_{min} - \frac{2|\psi|_{max}}{\left|r\,D\left(\frac{\rho_0 W}{r}\right)\right|_{\min}} \le c_r \le W_{max}.$$

5. INSTABILITY REGION

Theorem 5.1: *The range of complex eigen value* (c_r, c_i) *is given by*

$$\left[c_r-\left(\frac{W_{\max}+W_{\min}}{2}\right)\right]^2+c_i^2 \le \left(\frac{W_{\max}-W_{\min}}{2}\right)^2 - \frac{J_{\min}|DW|_{\min}}{2|DW|_{\max}^2\chi_{\min}+J_m+2|DW|_{\min}\sqrt{\chi_{\min}\left[J_m-k^2\right]}}.$$

Proof:

To prove the theorem, we adopt the method of Parthi & Nath (1991) [cf. 9].

Using $u=(W-c)G$ in (1) and (2), we have

$$D\left[(W-c)^2\rho_0\frac{|D(rG)|}{r}\right]-k^2\rho_0(W-c)^2G+\psi G=0, \tag{8}$$

with boundary conditions

$$G(R_1)= 0= G(R_1). \tag{9}$$

Multiplying (8) by (rG^*), integrating between R_1 and R_2, applying (9), we get

$$\int_{R_1}^{R_2}(W-c)^2\rho_0\left[\frac{|D(rG)|^2}{r}+k^2r|G|^2\right]dr - \int_{R_1}^{R_2}\psi|G|^2\,rdr=0 \tag{10}$$

$u=(W-c)G$, implies

$$G=\frac{u}{(W-c)}, \tag{11}$$

and $\dfrac{\left|D(rG)\right|^2}{r} \geq$

$$\frac{|W-c|^2\left|\dfrac{D(ru)}{r}\right|^2+|DW|^2|u|^2 r-2|W-c|\left|D(ru)\right||u||DW|}{|W-c|^4}. \qquad (12)$$

Substituting (11), (12) in (10) and using Cachy-Schwartz inequality, we get

$$\int_{R_1}^{R_2}\frac{\psi}{|W-c|^2}|u|^2\,rdr-k^2\int_{R_1}^{R_2}\rho_0|u|^2\,rdr$$

$$\geq(\rho_0)_{\min}\chi_{\min}\left[\int_{R_1}^{R_2}\rho_0|W-c|^2\left|\frac{D(ru)}{r}\right|^2+|DW|^2|u|^2 r\right]dr$$

$$-2\left[\int_{R_1}^{R_2}|W-c|^2\frac{\left|D(ru)\right|^2}{r}dr\right]^{1/2}\left[\int_{R_1}^{R_2}|DW|^2|u|^2\,rdr\right]^{1/2}, \qquad (13)$$

where, $\chi_{\min}=\dfrac{1}{|W-c|^2}$.

(13) can be written as

$\left[J_{\mathrm{m}}-k^2\right]E^2 \geq \chi_{\min}\left[D^2+B^2-2DB\right]$, where,

$$B^2=\int_{R_1}^{R_2}\rho_0|DW|^2|u|^2\,rdr, \qquad (14)$$

$$D^2=\int_{R_1}^{R_2}\rho_0|W-c|^2\left|\frac{D(ru)}{r}\right|^2 dr, \qquad (15)$$

$$E^2=\int_{R_1}^{R_2}\rho_0|DW|^2|u|^2\,rdr, \qquad (16)$$

and $J_m=\left[\dfrac{|\psi|}{|W-c|^2}\right]_{\max}$.

Solving for D and taking power 2 on both sides, we have

$$\chi_{\min}^2 D^2 \geq B^2\chi_{\min}^2+\chi_{\min}\left[J_{\mathrm{m}}-k^2\right]E^2 - 2\chi_{\min}B\sqrt{\chi_{\min}\left[J_{\mathrm{m}}-k^2\right]E^2} \qquad (17)$$

$$\int_{R_1}^{R_2}\rho_0\left[\frac{\left|D(rG)\right|^2}{r}+k^2r|G|^2\right]dr \geq$$

$$\int_{R_1}^{R_2}\left[\frac{|W-c|^2\dfrac{\left|D(ru)\right|^2}{r}+|DW|^2\dfrac{|ru|^2}{r}-2|W-c|\dfrac{\left|D(ru)\right|}{r}|ru||DW|}{|W-c|^4}+\frac{k^2r|u|^2}{|W-c|^2}\right]dr. \qquad (18)$$

Sub. (14), (15), (16) in the above equation, we get

$$\int_{R_1}^{R_2}\rho_0\left[\frac{\left|D(rG)\right|^2}{r}+k^2r|G|^2\right]dr \geq \chi_{\min}\left[D^2+B^2-2DB\right] +k^2\chi_{\min}E^2. \qquad (19)$$

Sub. (17) in (18), we get

$$\int_{R_1}^{R_2}\rho_0\left[\frac{\left|D(rG)\right|^2}{r}+k^2r|G|^2\right]dr$$

$$\geq\frac{1}{|W-c|^2|DW|^2}\left[\frac{2|DW|^2}{|W-c|^2}+J_m-2|DW|\sqrt{\chi_{\min}\left[J_{\mathrm{m}}-k^2\right]}\right]B^2. \qquad (20)$$

and $\displaystyle\int_{R_1}^{R_2}\psi r|G|^2\,dr \geq \frac{J_m\,B^2}{|W-c|^2}$. (21)

From Sasakura (1984) [cf. 10], we have

$$\left[c_r^2+c_i^2-(W_{\min}+W_{\max})c_r+W_{\min}W_{\max}\right]^2\int_{R_1}^{R_2}\rho_0\left[\frac{\left|D(rG)\right|^2}{r}+k^2r|G|^2\right]dr + \int_{R_1}^{R_2}\psi r|G|^2\,dr \leq 0. \qquad (22)$$

Sub. (20) and (21) in (22) , we have

$$\left[c_r-\left(\frac{W_{\max}+W_{\min}}{2}\right)\right]^2+c_i^{\,2} \leq \left(\frac{W_{\max}-W_{\min}}{2}\right)^2 - \frac{J_{\min}\,|DW|_{\min}}{2|DW|_{\max}^2\,\chi_{\min}+J_m-2|DW|_{\min}\sqrt{\chi_{\min}\left[J_{\mathrm{m}}-k^2\right]}}.$$

5.2 Theorem The range of complex eigen value (c_r, c_i) is given by

$$\left[c_r-\left(\frac{W_{\max}+W_{\min}}{2}\right)\right]^2+c_i^2 \le \left(\frac{W_{\max}-W_{\min}}{2}\right)^2$$
$$-\frac{\psi_{\min}}{\frac{2\pi^2 R_1}{R_2(R_2-R_1)^2}+J_m-\frac{2|W-c|\pi\sqrt{R_1}}{\sqrt{R_2}(R_2-R_1)}\sqrt{\chi_{\min}\left[J_m-k^2\right]}}.$$

Proof:

Using (16) and (19), we get

$$\int_{R_1}^{R_2}\rho_0\left[\frac{|D(rG)|^2}{r}+k^2r|G|^2\right]dr \ge$$
$$\frac{1}{|W-c|^2}\left[\frac{2\pi^2 R_1}{R_2(R_2-R_1)^2}+J_m-\frac{2|W-c|\pi\sqrt{R_1}}{\sqrt{R_2}(R_2-R_1)}\sqrt{\chi_{\min}\left[J_m-k^2\right]}\right]E^2. \tag{23}$$

And $$\int_{R_1}^{R_2}\rho_0\,\psi\, r|G|^2\,dr \ge \frac{\psi_m E^2}{|W-c|^2}. \tag{24}$$

Sub. (22), (23) in (21), we get

$$\left[c_r-\left(\frac{W_{\max}+W_{\min}}{2}\right)\right]^2+c_i^2 \le \left(\frac{W_{\max}-W_{\min}}{2}\right)^2$$
$$-\frac{\psi_{\min}}{\frac{2\pi^2 R_1}{R_2(R_2-R_1)^2}+J_m-\frac{2|W-c|\pi\sqrt{R_1}}{\sqrt{R_2}(R_2-R_1)}\sqrt{\chi_{\min}\left[J_m-k^2\right]}}.$$

6. CONCLUSION

In this paper, we study inviscid, incompressible, hydromagnetic swirling coaxial flows between rotating cylinders. We obtained upper bound for the growth rates of an unstable mode. Growth rate has been illustrated with three standard examples. Growth rate depends up on vorticity function, velocity profile and wave number. Graph shows that among the three standard examples Sinusoidal wave is the sharper among the other two waves. Furthermore, we obtained lower and upper bound for neutral phase velocity. We derived range of complex phase speed for growing perturbations. Also, we derived an instability regions depending on Rayleigh-Synge-Michael discriminant, velocity profile and radii. Solution of the heterogeneous equation for Couette flow model will be communicated later.

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