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In-Dominating Number and Out-Dominating Number of Unidirectional Perfect Binary Trees

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ABSTRACT: A diverging unidirectional perfect binary tree is a unidirectional perfect binary tree in which every internal node has exactly outdegree two and indegree 1 and all the leaf nodes are at the same level. A converging unidirectional perfect binary tree is a unidirectional perfect binary tree in which every internal node has exactly indegree two and outdegree 1 and all the leaf nodes are at the same level. In this paper, we study the out-domination number and in-domination number of the unidirectional perfect binary trees.

1. INTRODUCTION

A directed graph (digraph) is a pair $G_d = (V, E_d)$, where V is a set of nodes and $E_d \subseteq V \times V$ is a set of arcs, i.e., ordered pairs of nodes. We say that node u dominates node v (or node v is dominated by node u) if arc (u, v) is in G_d. Arc (u, v) is denoted as $u \rightarrow v$ graphically. In digraph G_d, the in-degree of node v is the number of arcs directed into v. The out-degree of node u is the number of arcs going out of u.

2. IN-DOMINATING SETS AND OUT-DOMINATING SETS

In digraph $G_d = (V, E_d)$, a set of nodes $D \subseteq V$ is a dominating set of G_d if each node $v \in V - D$ is dominated by at least a node in D. A minimal dominating set D_m is a dominating set with no proper subset of D_m as a dominating set. A minimum dominating set D_m is a dominating set of minimum cardinality. The cardinality of a minimum dominating set is called the domination number of G_d and is denoted by $\gamma(G_d)$.

Let $G_d = (V, E_d)$ be a directed graph with the set of nodes $V(G_d)$ and the set of arcs $E_d(G_d)$ such that each arc $(u, v) \epsilon E_d(G_d)$ is directed and u is said to be a predecessor of v, v is a successor of u, and u dominates v. Assume that G_d contains no loops or multiple arcs. For any node $v \in V$, the open in-neighborhood of v is the set $N^-(v) = \{u \in V: (u, v) \in E_d\}$ and the closed in-neighborhood of v is the set $N^-[v] = N^-(v) \cup \{v\}$. For a subset $\mathcal{A} \subseteq V$, the open in-neighborhood of $\mathcal{A}$ is $N^-(\mathcal{A}) = \bigcup_{v \in \mathcal{A}} N^-(v)$, and the closed in-neighborhood of S is

$N^-[\mathcal{A}] = \bigcup_{v\in\mathcal{A}} N^-[v]$. Analogously, for any node v, we define the open out-neighborhood of v is the set $N^+(v) = \{u \in V: (v,u) \in E_d\}$ and the closed out-neighborhood of v is the set $N^+[v] = N^+(v) \cup \{v\}$. For a set $\mathcal{A} \subseteq V$, the open out-neighborhood of S is $N^+(\mathcal{A}) = \bigcup_{v\in\mathcal{A}} N^+(v)$and the closed out-neighborhood of $\mathcal{A}$ is $N^+[v] = N^+(v) \cup \{v\}$.

A subset $\mathcal{A} \subseteq V$ is an in-dominating set of G_d, if $N^-[\mathcal{A}] = V$, that is, every node in V is dominated by at least one node in $\mathcal{A}$. The in-domination number $\gamma^-(G_d)$ is the minimum cardinality of an in-dominating set in G_d. A subset $\mathcal{A} \subseteq V$ is a out-dominating set of G_d, if $N^+[\mathcal{A}] = V$, or equivalently, every node in V is dominated by at least one node in $\mathcal{A}$. The out-domination number $\gamma^+(G_d)$ is the minimum cardinality of an out-dominating set in G_d.

Consider the below digraph G_d.

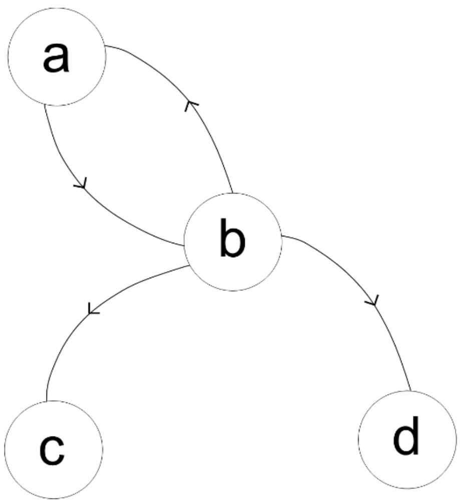

Figure 1: Digraph G_d

The only out-dominating set of cardinality 1 is $\{b\}$. So the number of out-dominating sets of minimal cardinality is 1. Hence the out-domination number $\gamma^+(G_d)$ is 1. But there are no in-dominating sets of cardinalities 1 and 2. The in-dominating sets of cardinality 3 are $\{a,c,d\}$, and $\{b,c,d\}$. So the number of minimal in-dominating sets 2. Hence the in-domination number $\gamma^-(G_d)$ is 2.

3. IN-DOMINATION NUMBER AND OUT-DOMINATION NUMBER OF SOME STANDARD UNIDIRECTIONAL TREES

3.1. Theorem 3.1

The out-domination number of diverging unidirectional perfect binary tree $\gamma^+(T_d)$ of height h with l_r nodes in r levels is

$$\gamma^+(T_d) = \begin{cases} \frac{4^{n+1}-1}{3}, & if\ height\ is\ odd \\ \frac{2(4^n)+1}{3}, & if\ height\ is\ even \end{cases}$$

Proof:

The below-structured tree is called diverging unidirectional perfect binary tree.

A diverging unidirectional perfect binary tree is a unidirectional perfect binary tree in which every internal node has exactly outdegree two and indegree 1 and all the leaf nodes are at the same level. Let *h* be the height of the tree. Here there are two cases for *h*, odd and even.

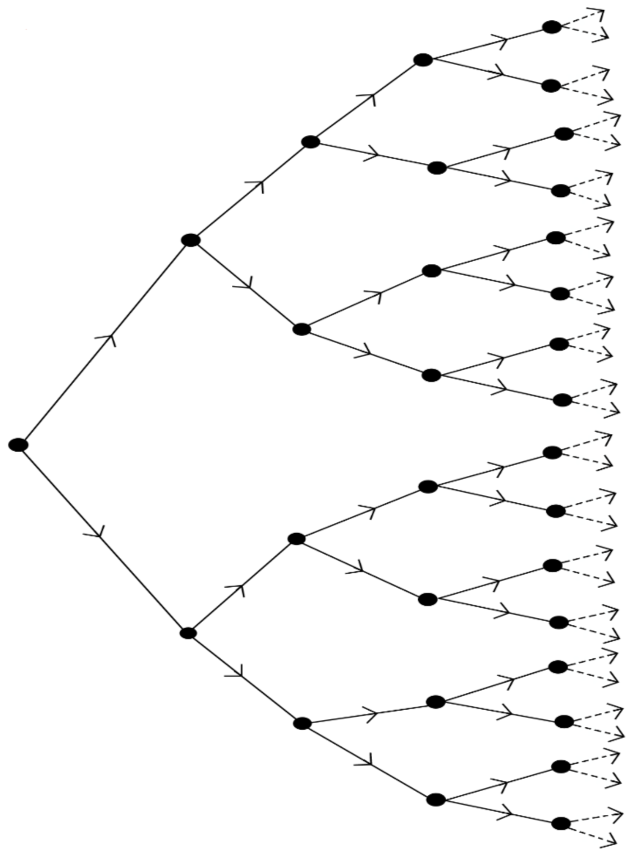

Figure 2:

If h is odd, then $h = 2n + 1$, where $n = 0, 1, 2, \ldots$ All the 2^{2n+1} terminal vertices in the $2n + 1$ level are out-dominated by all vertices 2^{2n} vertices in the $2n$ level. Similarly, all the 2^{2n-1} vertices in the $2n - 1$ level are out- dominated by all vertices 2^{2n-2} vertices in the $2n - 2$ level. Continuing this process, the one-level vertices are out-dominated by the single vertex in zero-level. Hence, the out-domination number of the unidirectional tree with odd height is

$$\gamma^+(T_d) = 1 + 2^2 + 2^4 + \cdots + 2^{2n} = \frac{4^{n+1} - 1}{3}$$

If h is even, then $h = 2n$, where $n = 0, 1, 2, \ldots$ All the 2^{2n} terminal vertices in $2n$ level are out-dominated by all 2^{2n-1} vertices in $2n - 1$ level. Similarly, all the 2^{2n-2} vertices in the $2n - 2$ level are out- dominated by all 2^{2n-3} terminal vertices in the $2n - 3$ level. Continuing this process the second-level vertices are out-dominated by the first-level 2 vertices and the zero-level vertex is dominated by itself. Hence the out-domination number of the tree with even height is

$$\gamma^+(T_d) = 1 + 2 + 2^3 + 2^5 + \cdots + 2^{2n-1}$$
$$= \frac{2(4^n)+1}{3}$$

The out-domination number of diverging unidirectional perfect binary tree of height h with l_r nodes in *r* levels is

$$\gamma^+(T_d) = \begin{cases} \frac{2(2^{2n+1}-1)}{3}, & if\ height\ is\ odd \\ \frac{2(4^n)+1}{3}, & if\ height\ is\ even \end{cases}$$

3.2. Theorem 3.2

The in-domination number of diverging unidirectional perfect binary tree of height h with l_r nodes in *r* levels is

$$\gamma^-(T_c) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ h\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ h\ is\ even \end{cases}$$

Proof:

If the height of the diverging unidirectional perfect binary tree is odd, then $h = 2n + 1$, where $n = 0, 1, 2, \ldots$ All the 2^{2n+1} terminal vertices in the $2n + 1$ level are in-dominated by itself and it dominates all vertices 2^{2n} vertices in the $2n$ level. Similarly, all the 2^{2n-2} vertices in the $2n - 2$ level are in-dominated by all vertices 2^{2n-1} vertices in the $2n - 1$ level. Continuing this process the last zero-level vertiex is in-dominated by the one-level vertices. Hence the in-domination number of the tree with odd height is

$$\gamma^-(T_c) = 2 + 2^3 + 2^5 + \cdots + 2^{2n+1}$$
$$= \frac{2(4^{n+1}-1)}{3}$$

If h is even, then $h = 2n$, where $n = 0, 1, 2, \ldots$ All the 2^{2n} terminal vertices in $2n$ level are in-dominated by itself and also it in-dominates all 2^{2n-1} vertices in the $2n - 1$ level. Similarly, all the 2^{2n-3} vertices in the $2n - 3$ level are in-dominated by all 2^{2n-2} vertices in the $2n - 2$ level. Repeating this process, the first-level vertices are in-dominated by the 2 vertices in the second-level and the zero-level vertex is dominated by itself. Hence the in-domination number of the tree with even height is

$$\gamma^-(T_c) = 1 + 2^2 + 2^4 + 2^6 + \cdots + 2^{2n}$$
$$= \frac{4^{n+1}-1}{3}$$

The in-domination number of diverging unidirectional perfect binary tree of height h with l_r nodes in *r* levels is

$$\gamma^-(T_c) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ h\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ h\ is\ even \end{cases}$$

3.3. Theorem 3.3

The out-domination number of converging unidirectional perfect binary tree $\gamma^+(T_c)$ of height h with l_r nodes in *r* levels is

$$\gamma^+(T_c) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ height\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ height\ is\ even \end{cases}$$

Proof:

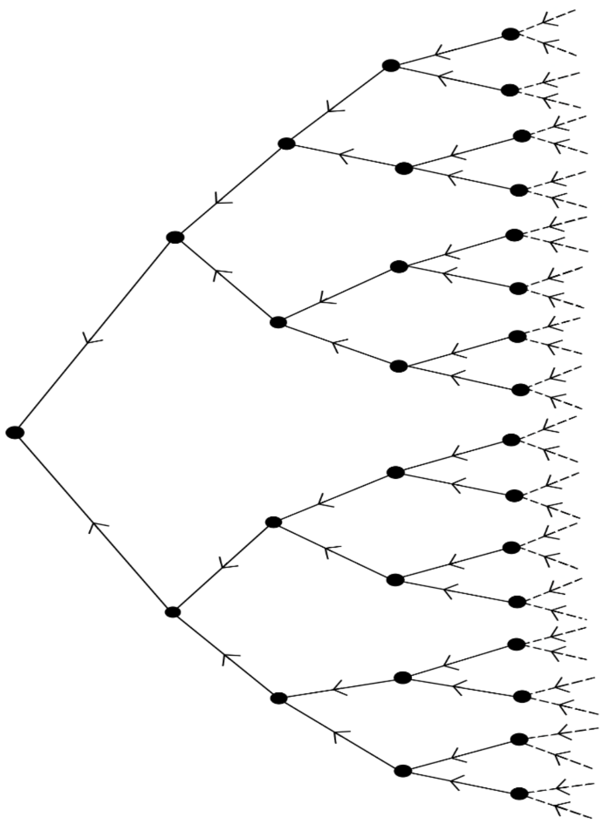

Figure 3:

The above-structured tree is called converging unidirectional perfect binary tree.

A converging unidirectional perfect binary tree is a unidirectional perfect binary tree in which every internal node has exactly indegree two and outdegree 1 and all the leaf nodes are at the same level. Let *h* be the height of the tree. Here there are two cases for *h*, odd and even.

If h is odd, then $h = 2n + 1$, where $n = 0, 1, 2, \dots$ All the 2^{2n+1} terminal vertices in the $2n + 1$ level are out-dominated by themselves and out-dominates all the 2^{2n} vertices in the $2n$ level. Similarly, all the 2^{2n-1} vertices in the $2n - 1$ level out-dominates all 2^{2n-2} vertices in the $2n - 2$ level. Proceeding like this the two one-level vertices dominates the single vertex in zero-level. Hence the out-domination number of the converging perfect binary tree with odd height is

$$\gamma^+(T_c) = 2 + 2^3 + 2^5 + \cdots + 2^{2n+1}$$
$$= \frac{2(4^{n+1}-1)}{3}$$

If h is even, then $h = 2n$, where $n = 0, 1, 2, \dots$ All the 2^{2n} terminal vertices in $2n$ level out-dominates by itself and it out-dominates all 2^{2n-1} vertices in $2n - 1$ level. Similarly, all the 2^{2n-2} vertices in the $2n - 2$ level out-dominates all 2^{2n-3} vertices in the $2n - 3$ level. Proceeding like this, the second-level 4 vertices out-dominates the first-level 2 vertices and the zero-level vertex is dominated by itself. Hence the out-domination number of the tree with even height is

$$\gamma^+(T_c) = 1 + 2^2 + 2^4 + \cdots + 2^{2n}$$
$$= \frac{4^{n+1}-1}{3}$$

The out-domination number of converging unidirectional perfect binary tree of height h with l_r nodes in *r* levels is

$$\gamma^+(T_c) = \begin{cases} \frac{2(4^{n+1}-1)}{3}, & if\ height\ is\ odd \\ \frac{4^{n+1}-1}{3}, & if\ height\ is\ even \end{cases}$$

Theorem 3.4

The in-domination number of converging unidirectional perfect binary tree of height h with l_r nodes in *r* levels is

$$\gamma^-(T_d) = \begin{cases} \frac{4^{n+1}-1}{3}, & if\ height\ is\ odd \\ \frac{2(4^n)+1}{3}, & if\ height\ is\ even \end{cases}$$

Proof:

If the height of the converging unidirectional perfect binary tree is odd, then $h = 2n + 1$, where $n = 0, 1, 2, \dots$ All the 2^{2n} vertices in the $2n$ level in-dominates 2^{2n+1} terminal vertices in the $2n + 1$ level. Similarly, all the 2^{2n-2} vertices in the $2n - 2$ level in-dominates all vertices 2^{2n-1} vertices in the $2n - 1$ level. Continuing this process, the 4 vertices in two-level in-dominates two vertices in one-level. And only one vertex in zero level is added to the in-dominating set. Hence the in-domination number of the tree with odd height is

$$\gamma^-(T_c) = 1 + 2^2 + 2^4 + \cdots + 2^{2n}$$
$$= \frac{4^{n+1}-1}{3}$$

If h is even, then $h = 2n$, where $n = 0, 1, 2, \dots$ All the 2^{2n} terminal vertices in $2n$ level are in-dominated by itself and also it in-dominates all 2^{2n-1} vertices in the $2n - 1$ level. Similarly, all the 2^{2n-3} vertices in the $2n - 3$ level are in-dominated by all 2^{2n-2} vertices in the $2n - 2$ level. Repeating this process, the zero-level vertex is in-dominated by the 2 first-level vertices. Hence the in-domination number of the tree with even height is

$$\gamma^{-}(T_c) = 2 + 2^3 + 2^5 + \cdots + 2^{2n-1}$$
$$= \frac{2(4^n)+1}{3}$$

The in-domination number of diverging unidirectional perfect binary tree of height h with l_r nodes in r levels is

$$\gamma^{-}(T_c) = \begin{cases} \frac{4^{n+1}-1}{3}, & if\ h\ is\ odd \\ \frac{2(4^n)+1}{3}, & if\ h\ is\ even \end{cases}$$

4. CONCLUSION AND FUTURE ENHANCEMENTS

In this paper, the authors have derived the In-domination number and out-domination number of the unidirectional perfect binary tree. Presently, the authors are working on the same concept with different types of unidirected tree graphs. In addition they are planning to find the minimal number of active nodes for the whole system to be active in selective patterns.

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